MTH 166

Lecture-21

Solution of Heat Equation

Topic:

Solution of Partial Differential Equations

Learning Outcomes:

To solve one dimensional Heat Equation

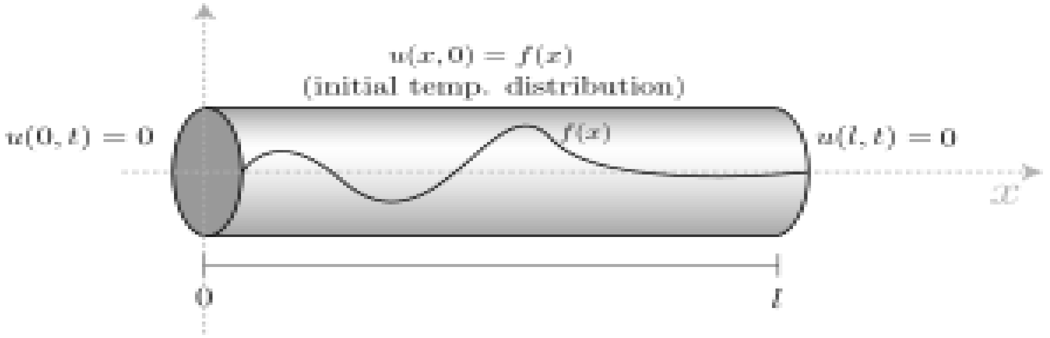
<u>Problem.</u> **Solve** $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Heat Equation];** c^2 **is Diffusivity constant.**

<u>Solution.</u> The given one dimensional wave equation is:

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)\ 0 \le x \le l\ (length), t > 0\ (time)$$

Let solution be: $u(x,t) = XT$ (2) where $X = f(x), T = g(t)$

$$\Rightarrow \frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$



Equation (1) becomes: $XT' = C^2 X''T$

$$\Rightarrow \frac{X''}{X} = \frac{1}{C^2}\frac{T'}{T} = k \quad \text{(Say)}$$

As k can take three values: zero, positive or negative, so we have following three cases.

<u>Case 1.</u> When $k = 0$

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = 0 \qquad \Rightarrow X'' = 0 \qquad \Rightarrow X = ax + b$$

Also $\frac{1}{C^2}\frac{T'}{T} = k \qquad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = 0 \qquad \Rightarrow T' = 0 \qquad \Rightarrow T = c$

Required solution of equation (1) is:

$$u(x,t) = XT = (ax + b)c = Ax + B$$

<u>Case 2.</u> When $k = p^2$ (Positive)

$$\frac{X''}{X} = k \qquad \Rightarrow \frac{X''}{X} = p^2 \qquad \Rightarrow X'' - p^2X = 0$$

<u>S.F.</u> $(D^2 - p^2)X = 0$

<u>A.E.</u> $(D^2 - p^2) = 0 \qquad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $\frac{1}{C^2}\frac{T'}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = p^2 \quad \Rightarrow T' - C^2p^2T = 0$

<u>S.F.</u> $(D - C^2p^2)T = 0$

<u>A.E.</u> $(D - C^2p^2) = 0 \quad \Rightarrow D = C^2p^2$

$\therefore T = ce^{C^2p^2t}$

Required solution of equation (1) is:

$u(x,t) = XT = (ae^{px} + be^{-px})\left(ce^{C^2p^2t}\right) = (Ae^{px} + \mathrm{B}e^{-px})e^{C^2p^2t}$

<u>**Case 3.**</u> When $k = -p^2$ (Negative)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2X = 0$

S.F. $(D^2 + p^2)X = 0$

A.E. $(D^2 + p^2) = 0 \qquad \Rightarrow D = \pm ip$

$\therefore X = e^{0x}(a\cos px + b\sin px)$

Also $\quad \frac{1}{C^2}\frac{T'}{T} = k \qquad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = -p^2 \qquad \Rightarrow T' + C^2p^2T = 0$

S.F. $(D + C^2p^2)T = 0$

A.E. $(D + C^2p^2) = 0 \qquad \Rightarrow D = -C^2p^2$

$\therefore T = ce^{-C^2p^2t}$

Required solution of equation (1) is:

$$u(x,t) = XT = (a\cos px + b\sin px)\left(ce^{-C^2p^2t}\right) = (A\cos px + \mathrm{B}\sin px)\, e^{-C^2p^2t}$$

This is the most suitable and practically feasible solution of wave equation.

Note: For Heat Equation

Equation: $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

Nature: Parabolic

Solution: 1.$u(x,t) = Ax + B$

2. $u(x,t) = (Ae^{px} + \mathrm{B}e^{-px})e^{C^2p^2t}$

3. $u(x,t) = (A\cos px + \mathrm{B}\sin px)\, e^{-C^2p^2t}$ (Most suitable one)